Sustainability of Ponzi scheme investment funds

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By: Artyom Kosyan

Supervisor: Gayane Barseghyan

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Abstract

A mathematical model is used to describe the behaviour of an investment fund that promises more return than it actually can deliver, also known as a Ponzi scheme. The model uses the following parameters: nominal interest rate, promised interest rate, investors' withdrawal rate, forthcoming investors rate, initial deposit and capital in order to estimate the money available in the fund over time and to predict for the fund whether it is sustainable or will collapse after a while. Different examples of parameters are taken to illustrate several cases of the dynamics of the investment fund. In addition, intuitive understanding of the roles of parameters in the model is discussed.

Keywords: Ponzi scheme, interest rate, withdrawal, sustainability

Acknowledgment

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1 Introduction

A very famous Italian swindler Charles Ponzi is known for his money-making schemes in early 1920s. He promised 50% return within 45 days or 100% in three months. He had convinced his clients that he bought international reply coupons (IRC) in other countries at a discount and sold them in the United States by their face values. In February 1920 his total take was $5,000 ($1 in 1920 was roughly $12 nowadays), in March 1920 the fund was already $25,000. After two more months in May 1920 the total budget was around $420,000. In June it reached to 25 million dollars and later in July he was filling the budget with a million dollars per day. However, his stories about the IRC were fake since the circulation of money in his fund required 160 million IRC coupons whereas there were only 27,000 in the market. His actual scheme was just redistribution of money of investors from new depositors to old ones and since the growth rate of new depositors was extremely high he was able to pay its old investors keeping the fund running. Eventually, after a year he was exposed and sent to a jail. A similar recent case was discovered in US. A man named Bernard Madoff made the same scheme and brought 18 billion dollars loss to its' clients. He promised around 10% return which was extremely high compared to the nominal interest rate at that time, which was around 3%. He was arrested in 2008, however nobody knows when exactly he started his criminal scheme but some estimations show that it has around 20 years of life. How fast must new deposits enter to the fund in order to sustain the fund? How much the investors must be willing to withdraw from their accounts in order not to pressure the fund's sustainability? In this paper we create a simple model of how to estimate the sustainability of this types of funds depending on different features.

2 Notations

For the mathematical modeling of the fund we will use the following notations

K - initial deposit

$s(t)$ - continuous cash inflow

r_p - promised rate of return

r_n - nominal interest rate

r_w - withdrawal rate of investors

r_i - investment rate

$W(t)$ - the total withdrawal at time t

$S(t)$ - the amount of money in the fund

3 Modeling the scheme

3.1 Withdrawals from the fund

We assume that the fund starts at time $t = 0$ with an initial deposit of $K \geq 0$. If the promised rate r_p is less then or equal to r_n then the fund is considered legal since the fund will generate money with a rate of $r_n - r_p$ and will give the return r_p back to its' customers. Otherwise, if $r_p \geq r_n$, then the fund promises a higher rate than it actually can deliver. This implies that the fund is not legal and even if it returns back the promised returns at the beginning, then for sure at some point it will not be able to satisfy all of its' investors. In this case the rate r_p is called the "Ponzi rate".

It is natural that investors will be willing to withdraw some part of the return leaving the rest in the fund for further growth. For simplicity we will assume that investors withdraw the money at a constant rate r_w at any time t applied to the promised accumulated return. Those investors who contributed

to the initial starting capital of K will have withdrawn $r_w K e^{t(r_p - r_w)}$ at time t. From the equation it is obvious that if the withdrawal rate r_w is lower than the promised r_p, then withdrawals increase exponentially since more return is added to the capital than withdrawn. The same way if $r_w \geq r_p$ then withdrawals will decrease exponentially.

Next we assume that besides the initial depositors there are further depositors who invest $s(v)$ at any time v. Those who invested at time v will withdraw their money starting from $t > v$ with the same withdrawal rate r_w. For a given v the money withdrawn at time v will similarly be $r_w s(v) e^{(t-v)(r_p - r_w)}$.

The total withdrawals $W(t)$ at time t will consist of total withdrawals from the initial deposit K and continuous withdrawals of cash inflow $s(v)$. The latter will be obtained by integrating the quantity $r_w s(v) e^{(t-v)(r_p - r_w)}$ obtained above from 0 to t.

$$W(t) = r_w K e^{t(r_p - r_w)} + r_w \int_0^t s(v) e^{(t-v)(r_p - r_w)} dv \qquad (1)$$

We can notice that only the promised rate and withdrawal rate play role in total withdrawals. The nominal interest rate does not appear in the formula.

Let's consider a case when each investor in average brings a certain number of new investors to the fund by telling them about the high returns. For simplicity let's consider that each investor contributes to the fund equally. In this scenario the continuous cash inflow $s(v)$ will have an exponential growth. This implies that we can model $s(v)$ the following way (Artzrouni 2009):

$$s(v) = s_0 e^{r_i v} \qquad (2)$$

Substituting the quantity obtained in (2) into withdrawal equation (1) we will get the following equation.

$$W(t) = r_w K e^{t(r_p - r_w)} + r_w \int_0^t s_0 e^{r_i v} e^{(t-v)(r_p - r_w)} dv \tag{3}$$

Computing the definite integral in (3) we will obtain the final expression for the withdrawal function:

$$W(t) = r_w e^{t(r_p - r_w)} \left(K + s_0 \frac{e^{t(r_w + r_i - r_p)} - 1}{(r_w + r_i - r_p)} \right) \tag{4}$$

3.2 Modeling the amount in the fund $S(t)$

We will use the function $S(t)$ to represent the total amount of money available in the fund at time t. Now let's see what will happen to $S(t)$ after a short period of time Δt. $S(t + \Delta t)$ is obtained by adding to $S(t)$ the nominal interest of $S(t)$ with the rate r_n in a short period of time Δt, the cash inflow $s(t)$ for Δt period and subtracting the withdrawals $W(t)$ again in a short period Δt:

$$S(t + \Delta t) = S(t) + S(t) r_n \Delta t + s(t) \Delta t - W(t) \Delta t$$

Modifying the equation we will get

$$\frac{S(t + \Delta t) - S(t)}{\Delta t} = S(t) r_n + s(t) - W(t)$$

For a very short period of Δt

$$\lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t} = S'(t)$$

which will lead us to the main differential equation the solution of which will be the amount of money in the fund at time t

$$S'(t) = S(t)r_n + s(t) - W(t) \qquad (5)$$

Substituting the values obtained in the equations (2) and (4) for $s(t)$ and $W(t)$ in (5) we will get the following:

$$S'(t) = S(t)r_n + s_0 e^{r_i t} - r_w e^{t(r_p - r_w)} \left(K + s_0 \frac{e^{t(r_w + r_i - r_p)} - 1}{(r_w + r_i - r_p)} \right)$$

Multiplying the entire equation with $e^{-r_n t}$ we will obtain

$$e^{-r_n t} S'(t) - e^{-r_n t} S(t)r_n = e^{-r_n t} \left[s_0 e^{r_i t} - r_w e^{t(r_p - r_w)} \left(K + s_0 \frac{e^{t(r_w + r_i - r_p)} - 1}{(r_w + r_i - r_p)} \right) \right]$$

Further modifying the equation we will get

$$e^{-r_n t} S'(t) - e^{-r_n t} S(t)r_n = e^{(r_i - r_n)t} s_0 \frac{r_i - r_p}{r_w + r_i - r_p} + e^{(r_p - r_w - r_n)t} r_w \frac{s_0 - K(r_w + r_i - r_p)}{r_w + r_i - r_p}$$

We can notice that the left part of the equation is the derivative of $e^{-r_n t} S(t)$. Integrating both sides of the equation with respect to t we will receive the following equation:

$$e^{-r_n t} S(t) = e^{(r_i - r_n)t} s_0 \frac{r_i - r_p}{(r_w + r_i - r_p)(r_i - r_n)} + e^{(r_p - r_w - r_n)t} r_w \frac{s_0 - K(r_w + r_i - r_p)}{(r_w + r_i - r_p)(r_p - r_w - r_n)} + \alpha$$

where α is a constant number. In the section **3.1** we stated that the fund starts at time 0 with initial investment of K from depositors. It is logical to assume that the fund owners could also have contributed to the fund at the very beginning without requiring any interest. It is more like **in-house** investment which we will denote by K_{ih}. Now we got the entire amount C available in the

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fund at time $t = 0$ which coinsides with $S(0)$

$$C = K_{ih} + K$$

This will help us to find α in the equation above since by substituting $t = 0$ we will rest with only unknown α .

$$C = \frac{s_0(r_n - r_p) + Kr_w(r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_p)} + \alpha$$

In order to write a compact equation for $S(t)$ let's denote

$$a = s_0 \frac{r_i - r_p}{(r_w + r_i - r_p)(r_i - r_n)}$$

$$m = r_i - r_n$$

$$b = r_w \frac{s_0 - K(r_w + r_i - r_p)}{(r_w + r_i - r_p)(r_p - r_w - r_n)}$$

$$n = r_p - r_w - r_n$$

$$\alpha = C - \frac{s_0(r_n - r_p) + Kr_w(r_i - r_n)}{(r_i - r_n)(r_n - r_p + r_w)}$$

and we will get

$$S(t) = ae^{(m+r_n)t} + be^{(n+r_n)t} + \alpha e^{r_n t} \tag{6}$$

We can also represent $S(t)$ as $e^{r_n t} f(t, a, m, b, n, \alpha)$ where

$$f(t, a, m, b, n, \alpha) = ae^{mt} + be^{nt} + \alpha$$

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3.3   Examples with different values of parameters

The function $S(t)$ always starts from the point $(0, C)$, where C is a positive number, the amount in the fund at time $t = 0$. We will look at some examples of how the function $S(t)$ behaves depending on the variables that are incuded in the formula of $S(t)$, and will try to understand them conceptually.

In the *Figure1* we see that the promised rate r_p is higher then the nominal interest rate r_n which means we have a Ponzi scheme. Investors drop out 4 % of their returns however, the fund still increases. This is because of the strong wave of new investors which is explained by r_i. The flow of new investors make the fund sustainable.

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Plot[f, {x, 0, 45}]
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$$K - 50,000$$
$$s_0 - 400$$
$$r_p - 0.08$$
$$r_n - 0.07$$
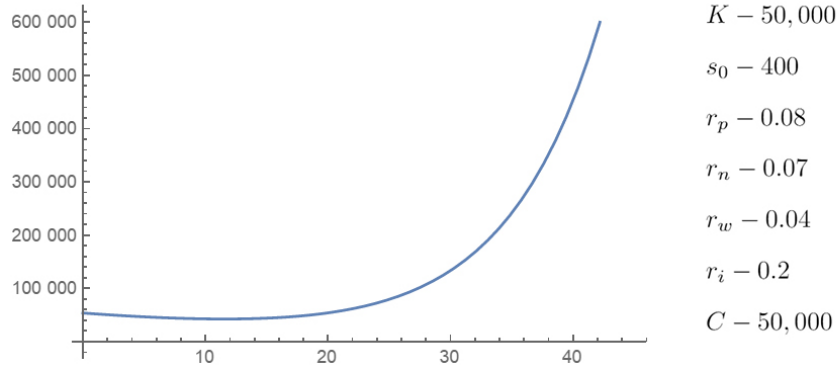$$r_w - 0.04$$
$$r_i - 0.2$$
$$C - 50,000$$

Figure 1

The problem with the example one is that the flow of new investors cannot stay that high all the time, since there are finite number of investors in the market. At some point the rate r_i of the flow of new investors will go down. Let's assume at some point r_i decreased to 0.01 which means there are still new depositors investing in the fund however they are not that many to sustain the fund. What will happen to the $S(t)$ graph in that case? *Figure2* illustrates the fund behaviour. After around 49 years the money in the fund will finish due to

the unrealistic high promised return r_p.

Plot[f, {x, 0, 60}]

$$K - 50,000$$
$$s_0 - 400$$
$$r_p - 0.08$$
$$r_n - 0.07$$
$$r_w - 0.04$$
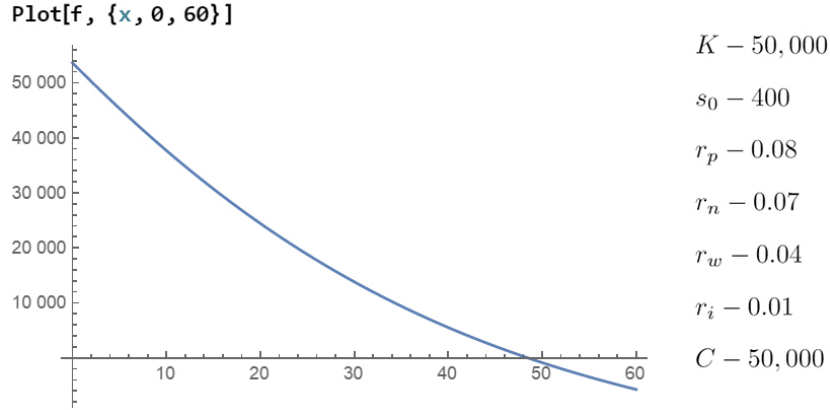$$r_i - 0.01$$
$$C - 50,000$$

Figure 2

In the next example (*Figure3*) we see that at some point the fund runs out of money. However, if fund managers borrow money to pay returns to investors acting like those returns are made on their invested money, they will recover after a while again raising the fund.
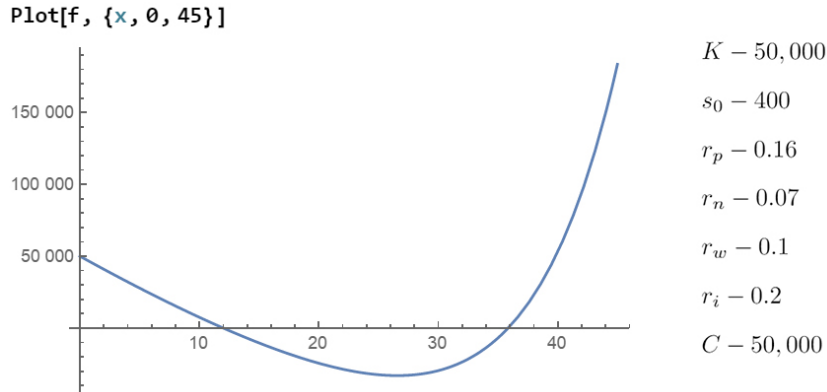
Plot[f, {x, 0, 45}]

$$K - 50,000$$
$$s_0 - 400$$
$$r_p - 0.16$$
$$r_n - 0.07$$
$$r_w - 0.1$$
$$r_i - 0.2$$
$$C - 50,000$$

Figure 3

Finally, let's assume the fund is not a "Ponzi" type, which means the promised return is less then the nominal interest rate. In this case the fund is legal and will never decrease since returns made by the fund managers is

greater then the returns paid back to investors. Moreover,the fund will even generate money with a rate of $r_n - r_p$. *Figure 4* illustrates the legal case of the fund management.
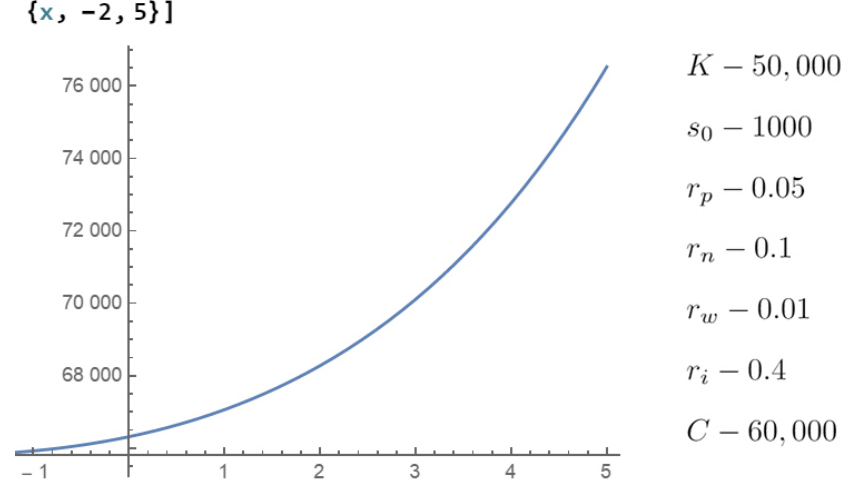
{x, -2, 5}]

$K - 50,000$

$s_0 - 1000$

$r_p - 0.05$

$r_n - 0.1$

$r_w - 0.01$

$r_i - 0.4$

$C - 60,000$

Figure 4

4   Further extension for $s(v)$ and $S(t)$

As we discussed in **3.1**, Mark Artzrouni suggested a constant groth rate for the new investments which we denoted by $s(v)$ at any time v. However, the rate of newcomers cannot always stay positive since at some point there will be lack of investors. Depending on time, the rate of change of the function $s(v)$ is positive at the beginning and then becomes negative tending to $-\infty$. For simplicity we can assume that the degree in the exponential equation is a quadratic equation in the form $4v - v^2$ which is positive initially and is negative after $v = 4$. The coefficient 4 is chosen in a way that $4v - v^2$ reaches the maximum value at the point $v = 2$, because based on different Ponzi schemes that have ever existed the average number of years that the funds' investment growth rates reach to their peaks is two years. This implies that we can model $s(v)$ the following way:

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$$s(v) = s_0 e^{r_i(4v - v^2)} \tag{7}$$

The graph bellow depicts the form of $s(v)$ which increases till the point $t = 2$ then starts decreasing tending to 0 when $v \to \infty$.
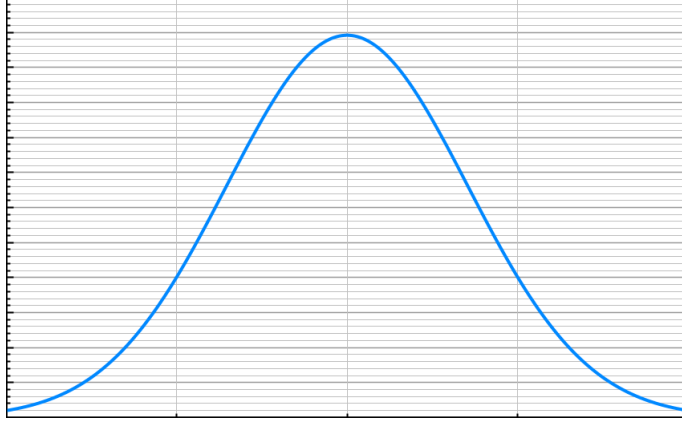

Figure 5: Investment function

Substituting the quantity obtained in (7) into withdrawal equation (1) we will get the following equation.

$$W(t) = r_w K e^{t(r_p - r_w)} + r_w \int_0^t s_0 e^{r_i(4v - v^2)} e^{(t-v)(r_p - r_w)} dv \tag{8}$$

We cannot represent the integral in the equation **(8)** with elementary functions because of the component e^{-v^2} in the under-integral expression. However, the integral $\int_0^t e^{-v^2} dv$ is easily computed with the help of Gauss error function $erf(t)$ which mostly occurs in probability, statistics and differential equations describing diffusion.

$$\int_0^t e^{-v^2} dv = \frac{\sqrt{\pi}}{2} erf(t)$$

Computing the definite integral in (8) we will obtain the final expression for the withdrawal function:

$$W(t) = r_w e^{t(r_p - r_w)} \left(K + s_0 \frac{\sqrt{\pi} e^{\frac{(r_i - r_p + r_w)^2}{2r_i}}}{2\sqrt{r_i}} \cdot erf\left(\frac{2r_i(t-2) + r_p - r_w}{2\sqrt{r_i}}\right)\right) \quad (9)$$

4.1 Modeling the fund capacity $S(t)$ using the new $s(v)$

The differential equation number (5) for the net amount of money in the fund doesn't change when we change $s(v)$. However, we will obtain a different solution to that equation in this section.

$$S'(t) = S(t)r_n + s(t) - W(t)$$

Applying the same steps as we did in the section **3.2** we will obtain the following equation

$$e^{-r_n t} S(t) = \int e^{-r_n t} \big(s(t) - W(t)\big) dt + const$$

Computing the integral in the right by substituting the functional forms of $s(t)$ and $W(t)$ we will obtain:

$$\int e^{-r_n t} s(t) dt = \int s_0 e^{r_i(4t - t^2)} dt = \frac{\sqrt{\pi} s_0 e^{\frac{(r_n - 4r_i)^2}{4r_i}}}{2\sqrt{r_i}} \cdot erf\left(\frac{2r_i(t-2) + r_n}{2\sqrt{r_i}}\right)$$

$$-\int e^{-r_n t} W(t) dt = \int r_w e^{t(-r_n + r_p - r_w)} \left(K + s_0 \frac{\sqrt{\pi} e^{\frac{(r_i - r_p + r_w)^2}{2r_i}}}{2\sqrt{r_i}} \cdot erf\left(\frac{2r_i(t-2) + r_p - r_w}{2\sqrt{r_i}}\right)\right) dt =$$

$$\frac{-r_w}{2(r_n + r_w - r_p)\sqrt{r_i}} \left(\sqrt{\pi} \cdot s_0 \cdot erf\left(\frac{r_n + 2r_i(t-2)}{2\sqrt{r_i}}\right) \cdot e^{\frac{r_n^2 - 8r_n r_i + r_p^2 - 2r_p r_w + 4r_p r_i + r_w^2 - 4r_w r_i + 2r_i^2}{4r_i}} - \right.$$

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$$-e^{t(-r_n+r_p-r_w)} \cdot \left(s_0\sqrt{\pi} e^{\frac{(r_i-r_p+r_w)^2}{2r_i}} \cdot erf\left(\frac{2r_i(t-2)+r_p-r_w}{2\sqrt{r_i}}\right) + 2K\sqrt{r_i}\right) + const$$

Therefore, $S(t)$ will be

$$S(t) = mA(t) + nB(t)e^{\lambda t} + \Theta \tag{10}$$

where m, n and λ are numbers obtained from the values of parameters, $A(t)$ and $B(t)$ are modified Gauss error functions.

$$A(t) = erf\left(\frac{r_n + 2r_i(t-2)}{2\sqrt{r_i}}\right)$$

$$B(t) = erf\left(\frac{2r_i(t-2)+r_p-r_w}{2\sqrt{r_i}}\right)$$

$$m = \frac{\sqrt{\pi}s_0 e^{\frac{(r_n-4r_i)^2}{4r_i}}}{2\sqrt{r_i}} - \frac{r_w}{2(r_n+r_w-r_p)\sqrt{r_i}}\left(\sqrt{\pi}\cdot s_0 \cdot e^{\frac{r_n^2-8r_n r_i + r_p^2 - 2r_p r_w + 4r_p r_i + r_w^2 - 4r_w r_i + 2r_i^2}{4r_i}}\right)$$

$$n = \frac{r_w}{2(r_n+r_w-r_p)\sqrt{r_i}}\cdot s_0\sqrt{\pi}e^{\frac{(r_i-r_p+r_w)^2}{2r_i}}$$

$$\lambda = -r_n + r_p - r_w$$

In the section 3.1 we stated that the fund starts at time 0 with initial investment of K from depositors. It is logical to assume that the fund owners could also have contributed to the fund at the very beginning without requiring any interest. It is more like in-house investment which we will denote by K_{ih}. Now we got the entire amount C available in the fund at time $t = 0$ which coinsides with $S(0)$. $C = K_{ih} + K$ This will help us to find Θ in the equation above since by substituting t = 0 we will rest with only unknown Θ.

$$\Theta = C - A(0) - B(0)$$

$A(0)$ and $B(0)$ are values of $A(t)$ and $B(t)$ at $t = 0$.

4.2 Roles and Relations of the parameters

Depending on the position of a parameter in the model equation, each of them has different power in making the scheme sustainable. We will investigate some parameters and relations of parameters intuitively in this section to see how they will affect our function $S(t)$.

1) $r_p \leq r_n$ relation simply means that the fund is not illegal and is making money just by investing in risk free assets such as T-bills.

2) $r_w \geq r_n$ relation means that if the schemer doesn't invest deposits in any market, then for sure at some point the scheme will collapse since $s(t) \to 0$ and investors withdraw more than the nominal interest rate fills the fund.

3)If $r_w \leq r_n$ all the time which means investors do not withdraw more than the nominal interest rate fills, then the scheme will never collapse, however, this is not reasonable since every person invests for future withdrawal. In average the fund withdrawals will exceed r_n taking the entire pool of investors.

4) r_i doesn't play any role in the fund's sustainability since $s(v)$ will always tend to zero because of $4t - t^2$ in the exponent. If r_i is high, then new investors will come a higher rate at the beginning, however, with the same logic after a while new investments will shrink rapidly.

5   Conclusion

After the Ponzi's scheme collapse, only within seven years it was possible to return 37 cents for a dollar invested by the clients. In case of Madoff, only several investors were able to recover their losses, whereas around 20 billion dollars out of 65 billion were stolen by Madoff, 27 billion were recovered and 18 billion were lost forever. The aim of this paper was to create a model that would predict sustainability of the Ponzi scheme fund depending mostly on the promised rate

of return, withdrawal rate, new investors entrance rate and nominal rate. Tiny changes in promised rate or people's behaviour about withdrawing their money can bring extremely big changes in the fund in a long run. For example, if withdrawal rate r_w is high but newcomers' rate r_i and nominal interest rate r_n are low then the fund is more likely to collapse after a while since fund managers will not be able to satisfy their clients by giving back their returns. If withdrawal rate is low but new investors come at a high rate and nominal interest rate is high too then the fund is more likely to sustain. However, this rates are not fixed. For example in a long run r_i effect in the cash inflow function will be negative since the number of investors is finite and they cannot grow infinitely or r_w will eventually increase since at some point each investor would like to withdraw more money because of their age etc. By the second Model of $S(t)$ we obtained that eventually $s(v)$ tends to zero because circulating money is scarce and cannot inflow to the fund infinitely. Hence, if at some point people's withdrawals exceed the return of the fund raised by the nominal interest rate, which do actually in most of the cases, then the fund will collapse. Further research and analysis on this topic can be done by making withdrawal rate, nominal interest rate as variables depending on time, since people's behaviour and even a risk free interest rate changes over time.

Taking money from someone to pay another one is an old practice, however, to avoid from such swindles we should be realistic about the investment offers. For example if someone promises you 30% return yearly with almost zero volatility on your investment but the average market return on similar investments is 4% then it causes doubts about its legitimacy. With the model described in this paper it is somehow possible to predict the future of such investment opportunities.

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